The Debt Tax Shield

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A tax shield is the reduction in income taxes that results from taking an allowable deduction from taxable income. Because interest on debt is a tax-deductible expense then taking on debt creates a tax shield. The equation for the value of the debt tax shield in any given month is...

$$\text{DTS value} = \text{Enterprise value} \times \text{Ratio of debt to enterprise value} \times \text{Debt interest rate} \times \text{Income tax rate} \times \frac{1}{12} \tag{1}$$

In this white paper we will calculate the value of the debt tax shield given the projected future path of enterprise value. To that end we will use the following hypothetical problem...

Our Hypothetical Problem

We are tasked with calculating the value of the debt tax shield given the following model assumptions...

Table 1: Expected Enterprise Value

Year Number	Enterprise Value	Annual Change
0	1,928,183	–
10	5,336,433	10.72%
20	8,961,585	5.32%
30	12,763,142	3.60%
40	17,405,646	3.15%
50	23,477,735	3.04%
75	49,215,487	3.00%
100	103,050,119	3.00%

Table 2: Debt Tax Shield Assumptions

Description	Value
Ratio of debt to enterprise value	40.00%
Weighted-average cost of capital	10.00%
Debt interest rate	6.00%
Income tax rate	25.00%

We want to build a model to answer the following questions:

Question 1: What is the value of the debt tax shield in month 120?
Question 2: What is the cumulative value of the debt tax shield over the time interval $[0,\infty]$?

Modeling Expected Enterprise Value

We will assume that over the time interval $[0,T]$ the short-term unsustainable rates (return on assets, revenue growth rate, etc.) are transitioning to the long-term sustainable rates such that the change in the log of enterprise value is non-linear. Using the equation for a parabola the approximating equation for enterprise value at time $t \le T$ is...

$$\mathbb{E}\Bigg[E_t\Bigg] = a\,t^2 + b\,t + c \text{ ...when... } t \le T \tag{2}$$

We will assume that over the time interval $[T,\infty]$ the short-term unsustainable rates have fully transitioned to the long-term sustainable rates such that the change in the log of enterprise value is now linear. If the variable μ is the constant rate of change of enterprise value then the approximating equation for enterprise value at time $t > T$ is...

$$\mathbb{E}\Bigg[E_t\Bigg] = \mathbb{E}\Bigg[E_T\Bigg] \text{Exp}\Bigg\{\mu\Bigg(t-T\Bigg)\Bigg\} \text{ ...when... } t > T \tag{3}$$

We will make the following definitions...

$$f(x)=a\,x^2+b\,x+c\ \text{ ...and... }\ f(y)=a\,y^2+b\,y+c\ \text{ ...and... }\ f'(z)=a\,z^2+b\,z+c \tag{4}$$

Note that we can write the functions in Equation (4) above as the following matrix:vector product...

$$\mathbf{A}\,\vec{\mathbf{u}}=\vec{\mathbf{v}}\ \text{ ...where... }\ \mathbf{A}=\begin{bmatrix}x^2 & x & 1\\ y^2 & y & 1\\ z^2 & z & 1\end{bmatrix}\ \text{ ...and... }\ \vec{\mathbf{u}}=\begin{bmatrix}a\\ b\\ c\end{bmatrix}\ \text{ ...and... }\ \vec{\mathbf{v}}=\begin{bmatrix}f(x)\\ f(y)\\ f(z)\end{bmatrix} \tag{5}$$

We will define the function $f(x)$ to be enterprise value at time $t=0$, the function $f(y)$ to be enterprise value at time $t=\frac{1}{2}T$, and the function $f(z)$ to be enterprise value at time $t=T$. Using these definitions we can rewrite Equation (5) above as...

$$\mathbf{A}\,\vec{\mathbf{u}}=\vec{\mathbf{v}}\ \text{ ...where... }\ \mathbf{A}=\begin{bmatrix}0 & 0 & 1\\ \frac{1}{4}T^2 & \frac{1}{2}T & 1\\ T^2 & T & 1\end{bmatrix}\ \text{ ...and... }\ \vec{\mathbf{u}}=\begin{bmatrix}a\\ b\\ c\end{bmatrix}\ \text{ ...and... }\ \vec{\mathbf{v}}=\begin{bmatrix}E_0\\ E_{\frac{1}{2}T}\\ E_T\end{bmatrix} \tag{6}$$

Using Equation (6) above the parameters to our parabola (a, b and c) are...

$$\mathbf{A}^{-1}\,\vec{\mathbf{v}}=\vec{\mathbf{u}} \tag{7}$$

The Debt Tax Shield

We will define the variable V_0 to be the value of the debt tax shield at time zero, the variable θ to be the ratio of debt value to enterprise value, the variable α to be the debt interest rate, the variable τ to be the income tax rate, and the variable κ to be the discount rate. Using Equation (1) above the equation for the value of the debt tax shield at time zero is...

$$V_0=\int\limits_0^\infty \alpha\,\tau\,\theta\,\mathbb{E}\Big[E_t\Big]\,\mathrm{Exp}\Big\{-\kappa\,t\Big\}\delta t \tag{8}$$

Using Equations (2) and (3) above we can rewrite Equation (8) above as...

$$V_0=\alpha\,\tau\,\theta\left[\int\limits_0^T\Big(a\,t^2+b\,t+c\Big)\,\mathrm{Exp}\Big\{-\kappa\,t\Big\}\delta t+\mathbb{E}\Big[E_T\Big]\int\limits_T^\infty \mathrm{Exp}\Big\{\mu\Big(t-T\Big)\Big\}\,\mathrm{Exp}\Big\{-\kappa\,t\Big\}\,\delta t\right] \tag{9}$$

Using Appendix Equations (17) and (19) below the solution to Equation (9) above is...

$$\begin{aligned}&V_0=\alpha\,\tau\,\theta\Big(I_1+\mathbb{E}\Big[E_T\Big]\,I_2\Big)\ \text{ ...where...}\\ &I_1=-\mathrm{Exp}\Big\{-\kappa\,T\Big\}\Big(a\Big(\kappa^2T^2+2\kappa T+2\Big)\kappa^{-3}+b\Big(\kappa T+1\Big)\kappa^{-2}+c\kappa^{-1}\Big)+2a\kappa^{-3}+b\kappa^{-2}+c\kappa^{-1}\\ &I_2=-\mathrm{Exp}\Big\{-\kappa\,T\Big\}\Big(\mu-\kappa\Big)^{-1}\end{aligned} \tag{10}$$

The Answers To Our Hypothetical Question

Question 1: What is the value of the debt tax shield in month 120?

Using our model assumptions and Equation (8) above the value of the debt tax shield in month 120 is...

$$\text{Value}=\alpha\,\tau\,\theta\,\mathbb{E}\Big[E_3\Big]=0.0600\times0.2500\times0.4000\times5,336,433\times\frac{1}{12}=\$2,668 \tag{11}$$

Question 2: What is the cumulative value of the debt tax shield over the time interval $[0,\infty]$?

Step 1: Restate debt tax shield assumptions to continuous-time where applicable:

Symbol	Description	Value	Notes
θ	Ratio of debt to enterprise value	0.4000	
κ	Weighted-average cost of capital	0.0953	$\ln(1+0.1000)$
α	Debt interest rate	0.0583	$\ln(1+0.0600)$
τ	Income tax rate	0.2500	

Step 2: Determine the value of the variable T:

We want to set the value of T such that enterprise value is accurately estimated over the first leg of the enterprise value curve as presented in Table 1 above. The further that we move out on the curve the less relevant is enterprise value due to discounting at the cost of capital. We will set the value of this variable to be $T = 40$.

Step 3: Determine the value of the variable μ:

Per the enterprise value curve as presented in Table 1 above the long-term sustainable growth rate of enterprise value appears to be 3.00%. The value of the this variable is therefore $\mu = \ln(1 + 0.0300) = 0.0296$.

Step 4: Determine the value of the parabola parameters a, b and c:

Using Equations (6) and (7) and the parameters in Table (1) above

$$\begin{bmatrix} 0 & 0 & 1 \\ 400 & 20 & 1 \\ 1600 & 40 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1928183 \\ 8961585 \\ 17405646 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1763 \\ 316404 \\ 1928183 \end{bmatrix} \tag{12}$$

Step 5: Calculate the value of the integral I_1 in Appendix Equation (17) below...

$$\begin{aligned} I_1 &= -\mathrm{Exp}\left\{-\kappa\,T\right\}\left(a\left(\kappa^2T^2+2\kappa T+2\right)\kappa^{-3}+b\left(\kappa T+1\right)\kappa^{-2}+c\kappa^{-1}\right)+2a\kappa^{-3}+b\kappa^{-2}+c\kappa^{-1} \\ &= -\mathrm{Exp}\left\{-0.0953\times 40\right\}\Bigg[1,763\times\left(0.0953^2\times 40^2+2\times 0.0953\times 40+2\right)\times 0.0953^{-3} \\ &+316,404\times\left(0.0953\times 40+1\right)\times 0.0953^{-2}+1,928,183\times 0.0953^{-1}\Bigg]+2\times 1,763\times 0.0953^{-3}+316,404\times 0.0953^{-2}\Bigg] \\ &= 53,896,952 \end{aligned} \tag{13}$$

Step 6: Calculate the value of the integral I_2 in Appendix Equation (19) below...

$$I_2 = -\mathrm{Exp}\left\{-\kappa\,T\right\}\left(\mu-\kappa\right)^{-1} = -\mathrm{Exp}\left\{-0.0953\times 40\right\}\times\left(0.0296-0.0953\right)^{-1} = 0.3364 \tag{14}$$

Step 7: Calculate the value of the debt tax shield:

Using Equation (10) above the value of the debt tax shield at time zero is...

$$V_0 = \alpha\,\tau\,\theta\left(I_1+\mathbb{E}\Big[E_T\Big]\,I_2\right) = 0.0583\times 0.2500\times 0.4000\times\left(53,896,952+17,405,646\times 0.3364\right) = 348,400 \tag{15}$$

References

[1] Gary Schurman, *The Schurman Parabola - Part I,* November, 2014

[2] Gary Schurman, *The Schurman Parabola - Part II,* November, 2014

Appendix

A We will define the integral I_1 to be the following equation...

$$I_1 = \int_0^T \left(a\,t^2 + b\,t + c\right) \mathrm{Exp}\left\{-\kappa\,t\right\} \delta t \tag{16}$$

The solution to the integral in Equation (16) above is... [2]

$$I_1 = -\mathrm{Exp}\left\{-\kappa\,T\right\}\left(a\left(\kappa^2 T^2 + 2\kappa T + 2\right)\kappa^{-3} + b\left(\kappa T + 1\right)\kappa^{-2} + c\kappa^{-1}\right) + 2a\kappa^{-3} + b\kappa^{-2} + c\kappa^{-1} \tag{17}$$

B We will define the integral I_2 to be the following equation...

$$I_2 = \int_T^\infty \mathrm{Exp}\left\{\mu\left(t - T\right) - \kappa\,t\right\} \delta t \tag{18}$$

The solution to the integral in Equation (18) above is... [2]

$$I_2 = -\mathrm{Exp}\left\{-\kappa\,T\right\}\left(\mu - \kappa\right)^{-1} \tag{19}$$